

LOCALIZABLE INVARIANTS OF CLOSED COMBINATORIAL MANIFOLDS AND EULER CHARACTERISTIC

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ABSTRACT. If a real value invariant of closed combinatorial manifolds admits a local formula that depends only on the \mathbf{f} -vector of the link of each vertex, then the invariant must be proportional to the Euler characteristic.

1. INTRODUCTION

Theorem 1.1 (Pachner 1986, [1]). *Two closed combinatorial n -manifolds are PL-homeomorphic if and only if it is possible to move between their triangulations using a sequence of bistellar moves (Pachner moves) and simplicial isomorphisms.*

This theorem suggests that we can look for quantities invariant under the bistellar moves so as to obtain invariants of closed combinatorial manifolds. The most famous one is the Euler characteristic defined as the alternating sum of numbers of simplices in a combinatorial manifold.

Theorem 1.2 (Roberts 2002, [2]). *Any linear combination of the numbers of simplices which is an invariant of closed combinatorial manifolds must be proportional to the Euler characteristic.*

A real value invariant Ψ of a closed combinatorial manifold M^n is called **localizable** if there exists a real value function ψ on the set of isomorphism classes of PL $(n-1)$ -spheres such that

$$\Psi(M^n) = \sum_{\text{vertex } v \in M^n} \psi(lk(v))$$

where $lk(v)$ is the link of a vertex v in M^n . We call ψ a **local formula** for Ψ .

Date: March 4, 2009.

2000 Mathematics Subject Classification. 57R20, 57Q99.

Key words and phrases. Combinatorial manifold, Local formula, Euler characteristic, PL sphere, Rational Pontryagin number.

This work is partially supported by a grant from NSFC (No.10826040).

Example 1.3. Euler characteristic $\chi(M^n)$ is localizable. A local formula for χ is:

$$\chi(M^n) = 1 + \sum_{k=0}^{n-1} (-1)^{k+1} \frac{f_k(lk(v))}{k+2}$$

where $f_k(lk(v))$ is the number of k -simplices in $lk(v)$.

Example 1.4. Rational Pontryagin numbers of closed combinatorial manifolds are localizable (see [3] and [4]). All known local formulae for rational Pontryagin numbers depend on the geometric patterns of the link of each vertex.

But could there exist any local formulae for rational Pontryagin numbers that only depend on the quantity of simplices in each link? More generally, we may ask the following.

Question 1: Can we find new localizable invariants of combinatorial manifolds other than Euler characteristic whose local formulae only depend on the quantity of simplices in the link of each vertex?

In this paper, we will give negative answer to the Question 1. Indeed, we will show the following.

Theorem 1.5. *Suppose Ψ is a localizable invariant of closed combinatorial manifolds M^n that can be written as*

$$\Psi(M^n) = \sum_{v \in M^n} \psi(f_0(lk(v)), \dots, f_{n-1}(lk(v)))$$

where $f_i(lk(v))$ is the number of i -simplices in the link $lk(v)$ of v in M^n , and ψ is an n -variable function on $f_0(lk(v)), \dots, f_{n-1}(lk(v))$. Then up to a constant, Ψ is proportional to the Euler characteristic.

Corollary 1.6. *There are no local formulae for rational Pontryagin numbers of closed combinatorial $4m$ -manifolds that depend only on the quantity of simplices in a link.*

Remark 1.7. In the category of all compact combinatorial manifolds (with or without boundary), it is shown in [5] that: a localizable invariant of compact combinatorial manifolds must be proportional to Euler characteristic. However, in the category of closed combinatorial manifolds, this statement is wrong because the rational Pontryagin numbers are localizable too.

The paper is organized as following. In Section 2, we discuss some combinatorial properties of bistellar moves which are useful in this paper. In Section 3, we define a type of PL n -disks in each dimension

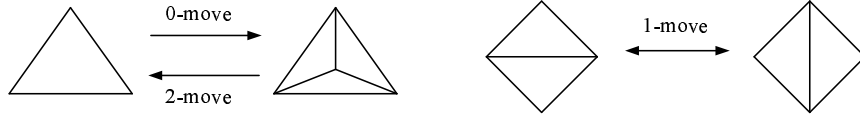


FIGURE 1. Bistellar moves in dimension 2

$n \geq 2$ (called gadget cells), which help us to understand how a local formula changes its values under all bistellar moves. In Section 4, we give a proof of Theorem 1.5. In Section 5, we do some simple calculations to verify Theorem (1.5) in dimension 4.

In addition, Theorem 1.5 is trivial in dimension 1. So we assume $n \geq 2$ in the rest of the paper. And we use C_n^k to denote the binomial coefficient $\binom{n}{k}$ through out the paper.

2. COMBINATORICS OF BISTELLAR MOVES

We first recall some definitions in combinatorial topology (see [6]).

Definition 2.1. Suppose X is a simplicial complex, the **star** $St(\sigma)$ of a simplex σ in X is the subcomplex consisting of all the simplices of X that contain σ . The **link** $lk(\sigma)$ of σ is the subcomplex consisting of all the simplices σ' of X with $\sigma' \cap \sigma = \emptyset$ and $\sigma' * \sigma$ being a simplex in X . A simplicial complex is called an n -dimensional **closed combinatorial manifold** if the link of each vertex of the complex is an $(n - 1)$ -dimensional PL sphere.

Definition 2.2. Let M^n be a combinatorial n -manifold and $\sigma \in M^n$ a $(n - i)$ -simplex ($0 \leq i \leq n$) such that its link in M^n is the boundary $\partial\tau$ of an i -simplex τ that is not a face of M^n . Then the operation

$$T_{\sigma, \tau}^{n, i}(M^n) := (M^n \setminus (\sigma * \partial\tau)) \cup (\partial\sigma * \tau)$$

is called a n -dimensional **bistellar i -move** (or Pachner move). Bistellar i -moves with $i \geq \lfloor \frac{n}{2} \rfloor$ is also called *reverse bistellar $(n - i)$ -move*.

For example: 0-move adds a new vertex to a triangulation (we assume that $\partial D^0 = \emptyset$), and a reverse 0-move deletes a vertex, see Figure 1 and Figure 2. Note that except 0-move and reverse 0-move, all other bistellar moves do not change the number of vertices in M^n .

When we apply a bistellar i -move to M^n , the links of some vertices involved in the move will be changed. We have the following simple observation.

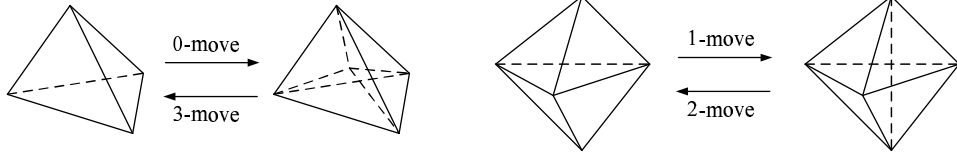


FIGURE 2. Bistellar moves in dimension 3

Lemma 2.3. *For any $0 < i < n$, doing an n -dimensional bistellar i -move $T_{\sigma,\tau}^{n,i}$ in M^n will induce an $(n-1)$ -dimensional bistellar i -move on the link of each vertex of σ and induce an $(n-1)$ -dimensional bistellar $(i-1)$ -move on the link of each vertex of τ . Doing an n -dimensional bistellar 0-move will induce an $(n-1)$ -dimensional bistellar 0-move on the link of each vertex of σ .*

Proof. For each vertex v_0 of a $(n-i)$ -simplex σ , let $\sigma \setminus \{v_0\}$ represent the codimension 1 face of σ that does not contain v_0 . Then the change of $lk(v_0)$ under the bistellar i -move $T_{\sigma,\tau}^{n,i}$ is:

$$\sigma \setminus \{v_0\} * \partial\tau \longrightarrow \partial(\sigma \setminus \{v_0\}) * \tau,$$

which by our notation is an $(n-1)$ -dimensional bistellar i -move $T_{\sigma \setminus \{v_0\}, \tau}^{n-1,i}$. Similarly, for any $u_0 \in \tau$, the change of $lk(u_0)$ under $T_{\sigma,\tau}^{n,i}$ is

$$\sigma * \partial(\tau \setminus \{u_0\}) \longrightarrow \partial\sigma * (\tau \setminus \{u_0\}),$$

which is an $(n-1)$ -dimensional bistellar $(i-1)$ -move $T_{\sigma, \tau \setminus \{u_0\}}^{n-1,i-1}$. \square

Let \mathcal{S}_n be the set of all isomorphism classes of $(n-1)$ -dimensional PL-spheres (without orientations). For any $L \in \mathcal{S}_n$, let $f_i(L)$ be the number of i -simplices in L and call $\mathbf{f}(L) = (f_0(L), \dots, f_{n-1}(L)) \in \mathbb{Z}_+^n$ the **f-vector** of L . In addition, we define $f_{-1}(L) := 1$.

Let

$$\mathcal{A}_n := \{\mathbf{f}(L) \in \mathbb{Z}_+^n \mid \forall L \in \mathcal{S}_n\}$$

For any $(n-1)$ -dimensional bistellar i -move $T^{n-1,i}$ on L , let $\beta^i(\mathbf{f}(L))$ be the **f-vector** of L after the move. It is easy to see that:

$$\beta^i \mathbf{f}(L) = (f_0(L) + r_{0,i}, \dots, f_{n-1}(L) + r_{n-1,i}), \quad (1)$$

where $r_{k,i} = C_{n-i}^{k-i} - C_{i+1}^{n-k}$. It is easy to check

$$r_{k,n-1-i} = -r_{k,i}, \quad \forall 0 \leq i, k \leq n-1 \quad (2)$$

$$\text{If } 2i = n-1, r_{ki} = 0, \quad \forall 0 \leq k \leq n-1 \quad (3)$$

So the reverse bistellar i -move on L gives

$$\beta^{n-1-i} \mathbf{f}(L) = (f_0(L) - r_{0,i}, \dots, f_{n-1}(L) - r_{n-1,i}),$$

Suppose Ψ is a localizable invariant of closed combinatorial manifolds and $\Psi(M^n) = \sum_{v \in M^n} \psi(\mathbf{f}(lk(v)))$ for some function ψ that only depends on $\mathbf{f}(lk(v))$. Then ψ is a function $\mathcal{A}_n \rightarrow \mathbb{R}$. By Theorem 1.1, Ψ is invariant under all bistellar moves. So for a bistellar i -move $T_{\sigma, \tau}^{n, i}$, according to Lemma 2.3, the function ψ must satisfy the following equations:

- When $i \neq 0$ or n , we have

$$\begin{aligned} \sum_{v \in \sigma} \psi(\beta^i \mathbf{f}(lk(v))) + \sum_{v' \in \tau} \psi(\beta^{i-1} \mathbf{f}(lk(v'))) &= \sum_{v \in \sigma} \psi(\mathbf{f}(v)) + \sum_{v' \in \tau} \psi(\mathbf{f}(v')) \\ \sum_{v \in \sigma} \psi(\beta^i \mathbf{f}(lk(v))) - \psi(\mathbf{f}(lk(v))) + \sum_{v' \in \tau} \psi(\beta^{i-1} \mathbf{f}(lk(v'))) - \psi(\mathbf{f}(lk(v'))) &= 0 \end{aligned} \quad (4)$$

- When $i = 0$, we have

$$\sum_{v \in \sigma} \psi(\beta^0 \mathbf{f}(lk(v))) - \psi(\mathbf{f}(v)) + \psi(\mathbf{f}_{\Delta^n}) = 0 \quad (5)$$

When $i = n$, we have

$$-\mathbf{f}_{\Delta^n} + \sum_{v' \in \tau} \psi(\beta^n \mathbf{f}(lk(v'))) - \psi(\mathbf{f}(v')) = 0 \quad (6)$$

where $\mathbf{f}_{\Delta^n} = (C_{n+1}^1, C_{n+1}^2, \dots, C_{n+1}^n)$ is the \mathbf{f} -vector of the boundary of an n -simplex.

Warning:

- (1) We can not conclude that $\psi(\beta^0 \mathbf{f}(lk(v))) - \psi(\mathbf{f}(v))$ is a constant directly from Equation (5). Because in general, we may not be able to guarantee all the vertices of σ have isomorphic links. Only when the PL n -ball is very symmetric, may we have this kind of configuration in the combinatorial structure. In general, we can not fit arbitrarily given PL n -balls together such that their intersection is a simplex.
- (2) It is not true that an arbitrary link $lk(v^*)$ could be involved in the Equation (4) for all i . Because when $i \geq 2$, there might be no bistellar i -move involving v^* . This means that for any $(n - i)$ -simplex or i -simplex σ in $St(v^*)$, either $lk(\sigma)$ is not isomorphic to the boundary of a simplex of complementary dimension, or $lk(\sigma)$ is the boundary of some existing simplex in the combinatorial manifold.

Remark 2.4. For any vertex v^* in a closed combinatorial manifold, v^* can always be involved in bistellar 0 and 1-moves.

3. HOW THE VALUES OF LOCAL FORMULA VARY UNDER BISTELLAR MOVES

In this section, we introduce some special type of PL n -disks in each dimension $n \geq 2$ and use them to figure out how a local formula ψ changes its value under bistellar moves if ψ depends only on the \mathbf{f} -vector of a link.

Lemma 3.1. *For each $n \geq 2$, there exists a PL n -disk K^n and a vertex $v_0 \in \partial K^n$ such that:*

- (1) ∂K^n is isomorphic to the boundary of an n -simplex.
- (2) for any $0 \leq i \leq n-1$, there exists a bistellar i -move $T_{\sigma,\tau}^{n,i}$ in K^n with $\sigma, \tau \subset K^n$ and $v_0 \in \sigma$, and $T_{\sigma,\tau}^{n,i}$ does not cause any changes to the star of any vertex on ∂K^n other than v_0 .

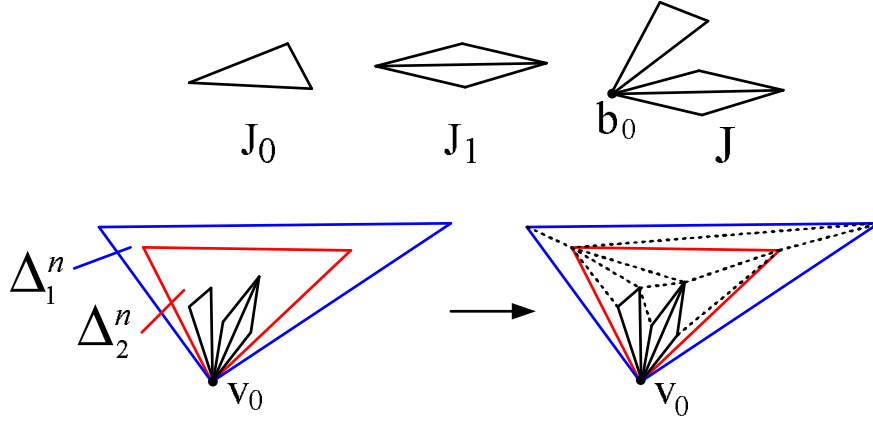
Proof. For each $0 \leq i \leq n-1$, let Δ^i be a simplex of dimension i . Let $J_i = \Delta^{n-i} * \partial \Delta^i$ and choose a vertex b_0^i of Δ^{n-i} in J_i . Let J be the one-point union of J_0, \dots, J_{n-1} got by gluing b_0^i to a base point b_0 . On the other hand, let Δ_1^n, Δ_2^n be two n -simplices such that $\Delta_2^n \subset \Delta_1^n$ and $\Delta_2^n \cap \partial \Delta_1^n$ is a vertex v_0 of both. Next, we glue b_0 to v_0 and put J inside Δ_2^n such that $J \cap \Delta_2^n = v_0$. By making up some new simplices and adding some new vertices inside Δ_2^n if necessary, we get a PL n -disk K^n (see Figure 3 for a construction of K^2). It is easy to see that doing the obvious bistellar i -move $T^{n,i}$ (replacing J_i by $\partial \Delta^{n-i} * \Delta^i$) associated to J_i inside Δ_2^n will not change the star of any vertex on ∂K^n except v_0 . So such a K^n satisfies all our requirements. \square

Note that our construction of K^n is far from unique. Here, we only need to choose one such K^n in each dimension $n \geq 2$. We call K^n the **gadget n -cell** and v_0 is called the **base point** of K^n . Let $a_{n,i}$ be the number of i -simplices of the link of v_0 in K^n that lie in the interior of Δ_1^n , and define $\mathbf{a}_n := (a_{n,0}, \dots, a_{n,n-1}) \in \mathbb{Z}^n$.

Let $\mathcal{A}'_n := \{\mathbf{f}(L) + \mathbf{a}_n \in \mathbb{Z}_+^n \mid \forall L \in \mathcal{S}_n\}$.

Lemma 3.2. $\mathcal{A}'_n \subset \mathcal{A}_n \subset \mathbb{Z}_+^n$.

Proof. For any $L \in \mathcal{S}_n$, let $U = v_0 * L$ be an PL n -ball. Obviously, the link of v_0 is isomorphic to L . Next, we choose an arbitrary n -simplex in U and turn it into the gadget n -cell K^n by subdivisions such that v_0 is the base point. Then the link of v_0 in U becomes a new PL $(n-1)$ -sphere whose \mathbf{f} -vector is $\mathbf{f}(L) + \mathbf{a}_n$. \square

FIGURE 3. A gadget cell K^2

Lemma 3.3. *Suppose ψ is local formula of a localizable invariant that only depends on the \mathbf{f} -vector of a link. For any $0 \leq i \leq n-1$, $\psi(\beta^i \mathbf{f}') - \psi(\mathbf{f}')$ is independent on $\mathbf{f}' \in \mathcal{A}'_n$.*

Proof. Suppose $L \in \mathcal{S}_n$ such that $\mathbf{f}(L) + \mathbf{a}_n = \mathbf{f}'$. Suppose v^* is a vertex in a closed combinatorial manifold M^n such that $lk(v^*) \cong L$. We choose an n -simplex in $St(v^*)$ and turn it into the gadget n -cell K^n by subdivisions such that v^* is the base point. Now, $\mathbf{f}(lk(v^*)) = \mathbf{f}(L) + \mathbf{a}_n = \mathbf{f}'$.

For any $0 \leq i \leq n-1$, We do the obvious n -dimensional bistellar i -move $T^{n,i}$ associated to J_i in the gadget cell. Let u_1^i, \dots, u_{n+1}^i be all the vertices involved in $T^{n,i}$ other than v^* . By the construction of K^n in Lemma 3.1, for each $1 \leq j \leq n+1$, the star of u_j^i is completely inside K^n and the change of $St(u_j^i)$ under $T^{n,i}$ is also inside K^n . So in the Equations (4) — (6), all terms are canonically decided by K^n except $\psi(\beta^i \mathbf{f}(lk(v^*))) - \psi(\mathbf{f}(lk(v^*)))$. So $\psi(\beta^i \mathbf{f}') - \psi(\mathbf{f}')$ does not depend on \mathbf{f}' . \square

For any $\mathbf{f}' \in \mathcal{A}'_n$ and any $0 \leq i \leq n-1$, let

$$\psi(\beta^i \mathbf{f}') - \psi(\mathbf{f}') := H_i^n \in \mathbb{R}^1$$

Lemma 3.4. *For any $0 \leq i \leq n-1$, we have*

- (1) $(n-i+1) \cdot H_i^n + (i+1) \cdot H_{i-1}^n = 0$, and $H_{-1}^n := \psi(\mathbf{f}_{\Delta^n})$,
 $H_n^n := -\psi(\mathbf{f}_{\Delta^n})$.
- (2) $H_i^n = -H_{n-1-i}^n$.
- (3) $H_i^n = 0$ if $2i = n-1$.

So each H_i^n is a rational multiple of $\psi(\mathbf{f}_{\Delta^n})$.

Proof. Take a bistellar i -move $T_{\sigma,\tau}^{n,i}$ in a closed combinatorial n -manifold. For each vertex v of σ and τ , we choose an n -simplex in $St(v)$ and turn it into the gadget cell K^n by subdivisions (of course, sharing gadget cells between different stars are allowed). Then each $\mathbf{f}(lk(v))$ becomes an element in \mathcal{A}'_n after these subdivisions. By the Equations (4) — (6) and Lemma 3.3, we have the first equality. The second and third equalities follow easily from the first one. \square

Lemma 3.5. *Suppose ψ is a local formula of a localizable invariant that only depends on the \mathbf{f} -vectors of links. For any $0 \leq i \leq n-1$ and any $(n-1)$ -dimensional bistellar i -moves on a PL $(n-1)$ -sphere L , $\psi(\beta^i \mathbf{f}(L)) - \psi(\mathbf{f}(L)) = H_i^n$.*

Proof. Suppose $T^{n-1,i}$ is an $(n-1)$ -dimensional bistellar i -move on $L \in \mathcal{S}^n$. Then we can find a vertex v^* in a closed combinatorial manifold M^n such that $lk(v^*) \cong L$ and an n -dimensional bistellar i -move $T_{\sigma,\tau}^{n,i}$ in M^n such that $v^* \in \sigma$, and the move $T^{n-1,i}$ on $lk(v^*)$ is induced from $T_{\sigma,\tau}^{n,i}$.

For any vertex $v \neq v^*$ of σ and τ , we can choose an n -simplex $\Delta_v^n \subset St(v)$ with $\Delta_v^n \not\subseteq St(v^*)$, and then turn Δ_v^n into the gadget cell K^n by subdivisions. After these subdivisions, $lk(v)$ is changed and $\mathbf{f}(lk(v))$ becomes an element in \mathcal{A}'_n while $St(v^*)$ stays the same. So by Lemma 3.3 and the Equation (4), we have the following.

$$\psi(\beta^i \mathbf{f}(lk(v^*))) - \psi(\mathbf{f}(lk(v^*))) + (n-i)H_i^n + (i+1)H_{i-1}^n = 0$$

By Lemma 3.4 (1), we have

$$\psi(\beta^i \mathbf{f}(L)) - \psi(\mathbf{f}(L)) = \psi(\beta^i \mathbf{f}(lk(v^*))) - \psi(\mathbf{f}(lk(v^*))) = H_i^n$$

\square

Remark 3.6. The main idea in the above lemmas is: the change of the \mathbf{f} -vector of a PL -sphere caused by different bistellar i -moves are the same, i.e. the change does not depend on where the i -moves take place. So the variations of $\psi(L)$ under different bistellar i -moves on a PL -sphere L are the same if the value of $\psi(L)$ only depends on the \mathbf{f} -vector of L .

4. PROOF OF THEOREM 1.5

For any PL $(n-1)$ -sphere L , by Theorem (1.1), we can apply a finite sequence of $(n-1)$ -dimensional bistellar moves to the boundary of an n -simplex Δ^n to get L . For each $0 \leq i \leq n-1$, suppose we

have $m_i(L)$ bistellar i -moves in the sequence. Then by Equation (1), $m_0(L), \dots, m_{n-1}(L)$ satisfy

$$\sum_{i=0}^{n-1} m_i(L) \cdot r_{k,i} = f_k(L) - C_{n+1}^{k+1}, \quad \forall 0 \leq k \leq n-1 \quad (7)$$

The solution to Equation (7) is not unique because adding a bistellar i -move and its reverse simultaneously to a solution will give a new solution. But the only thing important here is $m_i(L) - m_{n-1-i}(L)$ which is, in fact, uniquely determined by $\mathbf{f}(L)$.

Lemma 4.1. *For any $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, $L \in \mathcal{S}_n$, $m_i(L) - m_{n-1-i}(L)$ is uniquely determined by $f_0(L), \dots, f_{n-1}(L)$ in Equation (7).*

Proof. First of all, by Equation (2) and (3), the Equation (7) becomes:

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (m_i(L) - m_{n-1-i}(L)) \cdot r_{k,i} = f_k(L) - C_{n+1}^{k+1}, \quad \forall 0 \leq k \leq n-1 \quad (8)$$

In addition, the Dehn-Sommerville equations of PL -spheres imply that the $\mathbf{f}(L)$ is completely decided by $f_0(L), \dots, f_{\lfloor \frac{n}{2} \rfloor - 1}(L)$. So the solution to above system (8) of n linear equations equal the solution to the $\lfloor \frac{n}{2} \rfloor$ linear equations:

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (m_i(L) - m_{n-1-i}(L)) \cdot r_{k,i} = f_k(L) - C_{n+1}^{k+1}, \quad \forall 0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1. \quad (9)$$

Notice when $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, $r_{k,i} = C_{n-i}^{k-i}$. So

- if $k < i$, $r_{k,i} = 0$.
- if $k = i$, $r_{i,i} = 1$.

The square integral matrix $(r_{k,i})_{0 \leq k, i \leq \lfloor \frac{n}{2} \rfloor - 1}$ is invertible over \mathbb{Z} . So the above system of $\lfloor \frac{n}{2} \rfloor$ linear equations (9) has a unique solution. \square

Remark 4.2. When $n = 2s + 1$ is odd, by Equation (3), $r_{k,s} = 0$ for any $0 \leq k \leq n-1$. So in Equation (8), the term $m_s \cdot r_{k,s}$ is omitted. Indeed, we will see that when $n = 2s + 1$, the value of m_s does not affect our calculation of ψ since $\beta^s \mathbf{f} = \mathbf{f}$ and $H_s^n = 0$ (see Lemma 3.4).

For any $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ and $L \in \mathcal{S}_n$, we can assume

$$m_i(L) - m_{n-1-i}(L) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_{ik} (f_k(L) - C_{n+1}^{k+1}), \quad c_{ik} \in \mathbb{Z}$$

Proof of Theorem 1.5: For a local formula ψ of a real value localizable invariant Ψ of closed combinatorial n -manifolds, by Lemma 3.4 and Lemma 3.5, for any $L \in \mathcal{S}_n$, we have

$$\begin{aligned}
\psi(\mathbf{f}(L)) &= \psi(\mathbf{f}_{\Delta^n}) + \sum_{i=0}^{n-1} m_i(L) \cdot H_i^n \\
&= \psi(\mathbf{f}_{\Delta^n}) + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} (m_i(L) - m_{n-1-i}(L)) \cdot H_i^n \\
&= \psi(\mathbf{f}_{\Delta^n}) + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} H_i^n \cdot c_{ik} (f_k(L) - C_{n+1}^{k+1}). \quad (10)
\end{aligned}$$

So $\psi(\mathbf{f}(L))$ is a linear function of $f_0(L), \dots, f_{n-1}(L)$. Moreover, since each H_i^n is a rational multiple of $\psi(\mathbf{f}_{\Delta^n})$, so up to a constant factor $\psi(\mathbf{f}_{\Delta^n})$, we can write

$$\psi(\mathbf{f}(L)) = \sum_{k=-1}^{\lfloor \frac{n}{2} \rfloor - 1} b_k \cdot f_k(L), \quad b_k \in \mathbb{Q} \quad (\text{note } f_{-1}(L) = 1)$$

Then for a closed combinatorial n -manifold M^n ,

$$\begin{aligned}
\Psi(M^n) &= \sum_{v \in M^n} \psi(\mathbf{f}(lk(v))) \\
&= \sum_{v \in M^n} \sum_{k=-1}^{\lfloor \frac{n}{2} \rfloor - 1} b_k \cdot f_k(lk(v)) \quad (11)
\end{aligned}$$

For a closed combinatorial manifold M^n , let $f_k(M^n)$ be the number of k -simplices in M^n . Then obviously

$$f_k(M^n) = \frac{1}{k+1} \sum_{v \in M^n} f_{k-1}(lk(v)), \quad 0 \leq k \leq n$$

By equation (11),

$$\Psi(M^n) = \sum_{k=-1}^{\lfloor \frac{n}{2} \rfloor - 1} b_k (k+2) \cdot f_{k+1}(M^n)$$

So $\Psi(M^n)$ is a linear function of $f_0(M^n), \dots, f_n(M^n)$. Then by Theorem 1.2, Ψ must be proportional to the Euler characteristic. \square

5. VERIFICATION OF THEOREM 1.5 IN DIMENSION 4

When $n = 4$, by the Dehn-Sommerville equations for PL-spheres, we find that the \mathbf{f} -vector of a PL 3-sphere L depends only on the number of vertices and edges in L . So we can write

$$\psi(\mathbf{f}(L)) = \psi(f_0(L), f_1(L)), \quad \forall L \in \mathcal{S}_4$$

The system of equations (9) are:

$$\begin{aligned} m_0 - m_3 &= f_0(L) - 5, \\ 4(m_0 - m_3) + (m_1 - m_2) &= f_1(L) - 10 \end{aligned}$$

So $m_0 - m_3 = f_0(L) - 5$, $m_1 - m_2 = f_1(L) - 4f_0(L) + 10$. In addition, by Lemma (3.4), we have

$$H_0^4 = -\frac{1}{5}\psi(\mathbf{f}_{\Delta^4}); \quad H_1^4 = \frac{1}{10}\psi(\mathbf{f}_{\Delta^4}).$$

Then by Equation (10), we have

$$\begin{aligned} \psi(\mathbf{f}(L)) &= \psi(\mathbf{f}_{\Delta^4}) \left(1 - \frac{1}{5}(f_0(L) - 5) + \frac{1}{10}(f_1(L) - 4f_0(L) + 10) \right) \\ &= 3\psi(\mathbf{f}_{\Delta^4}) \left(1 - \frac{1}{5}f_0(L) + \frac{1}{30}f_1(L) \right). \end{aligned}$$

This is proportional to the local formula for Euler characteristic. In fact, a local formula for Euler characteristic of 4-dimensional closed combinatorial manifolds M^4 is: for any vertex $v \in M^4$,

$$\psi_\chi(lk(v)) = 1 - \frac{f_0(lk(v))}{2} + \frac{f_1(lk(v))}{3} - \frac{f_2(lk(v))}{4} + \frac{f_3(lk(v))}{5},$$

The Dehn-Sommerville equations for the PL 3-spheres imply:

$$f_2(lk(v)) = 2f_3(lk(v)), \quad f_3(lk(v)) = f_1(lk(v)) - f_0(lk(v)),$$

$$\text{so we get: } \psi_\chi(lk(v)) = 1 - \frac{f_0(lk(v))}{5} + \frac{f_1(lk(v))}{30}.$$

So $\psi(lk(v)) = 3\psi(\mathbf{f}_{\Delta^4}) \cdot \psi_\chi(lk(v))$. Then we have

$$\Psi(M^4) = 3\psi(\mathbf{f}_{\Delta^4}) \cdot \chi(M^4).$$

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